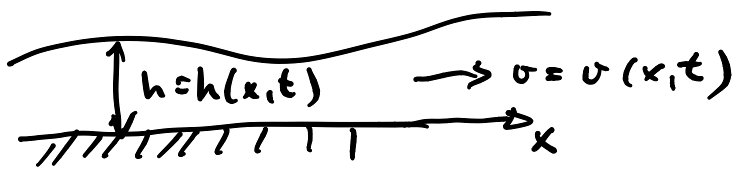
Ch 5

$$u_t + f(u)_x = 0$$

$u = u(x,t) \in \mathbb{R}^n$, $x \in \mathbb{R}$, $t \in [0,\infty)$

$f = f(u) = (f_1(u), \ldots, f_n(u))$

Shallow water equations



$$h_t + (hv)_x = 0 \quad \leftarrow \text{mass}$$

$$(hv)_t + \left(hv^2 + \tfrac{1}{2}h^2\right)_x = 0 \leftarrow \text{momentum}$$

Introduce $q = hv$. Then we have

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ q^2/h + \frac{1}{2}h^2 \end{pmatrix}_x = 0$$

$u = \begin{pmatrix} h \\ q \end{pmatrix}$, $f(u) = \begin{pmatrix} q \\ q^2/h + \frac{1}{2}h^2 \end{pmatrix}$ $\quad u_t + f(u)_x = 0$

Vacuum, i.e., $h=0$, is a serious problem.

Wave equation

$$\phi_{tt} = (c^2\phi_x)_x$$

Introduce $u = \phi_x$, $v = \phi_t$.

We get

$$u_t = \phi_{xt} = v_x$$
$$v_t = \phi_{tt} = (c^2\phi_x)_x = (c^2u)_x$$

or

$$\begin{pmatrix} u \\ v \end{pmatrix}_t - \begin{pmatrix} v \\ c^2u \end{pmatrix}_x = 0$$

p-system

A model for isentropic gas

v = specific volume = $1/\rho$

u = velocity

$p = p(v)$ pressure

One can show that

$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} -u \\ p(v) \end{pmatrix}_x = 0$$

o

Euler equations

Compressible fluid

ρ = density , v = velocity

p = pressure , E = energy

mass conservation : $\rho_t + (\rho v)_x = 0$

momentum conservation : $(\rho v)_t + (\rho v^2 + p)_x = 0$

energy conservation : $E = \frac{1}{2}\rho v^2 + \rho e$

where e = specific internal energy, a given function, $e = e(\rho, p)$

$$E_t + (v(E+p))_x = 0$$

Unknowns are ρ, v, p

$$u_t + f(u)_x = 0$$

$$u = (u_1, \dots, u_n) \in \mathbb{R}^n, \quad f = (f_1, \dots, f_n)$$

$$f(u)_x = df(u)\, u_x = f'(u)\, u_x$$

$$df(u) = n \times n \text{ matrix}$$

$$= \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{pmatrix}$$

The equation is <u>hyperbolic</u> if <u>df has n real eigenvalues</u>

$$df(u)\, r_j = \lambda_j r_j$$

$\lambda_1(u) \le \lambda_2(u) \le \dots \le \lambda_n(u)$ eigenvalues

$r_j(u)$ eigenvectors

The equation is strictly hyperbolic if the n eigenvalues are distinct, i.e.,

$$\lambda_1(u) < \lambda_2(u) < \dots < \lambda_n(u)$$

Shallow water eq'n:

$$\begin{pmatrix} h \\ q \end{pmatrix}_t + \begin{pmatrix} q \\ q^2/h + \frac{1}{2}h^2 \end{pmatrix}_x = 0$$

$$f(u) = \begin{pmatrix} q \\ q^2/h + 1/2\, h^2 \end{pmatrix}$$

$$df(u) = \begin{pmatrix} 0 & 1 \\ -\frac{q^2}{h^2} + h & \frac{2q}{h} \end{pmatrix}$$

$$|df(u) - \lambda| = \begin{vmatrix} -\lambda & \cancel{q}\ 1 \\ -\frac{q^2}{h^2} + h & 2\frac{q}{h} - \lambda \end{vmatrix}$$

$$= -\lambda\left(2\frac{q}{h} - \lambda\right) - \cancel{q}\,1\left(-\frac{q^2}{h^2} + h\right)$$

$$= \lambda^2 - 2\frac{q}{h}\lambda - \cancel{q}\,1\left(-\frac{q^2}{h^2} + h\right) = 0$$

$$\lambda = \frac{1}{2}\left[2\frac{q}{h} \pm \sqrt{\left(-2\frac{q}{h}\right)^2 - 4\left(-\frac{q^2}{h^2} + h\right)\left(-\cancel{q}\,1\right)}\ \right]$$

$$= \frac{q}{h} \pm \left(\left(\frac{q}{h}\right)^2 - \frac{q^2}{h^2} + h\right)^{1/2}$$

$$= \frac{q}{h} \pm \sqrt{h}$$

. Thus $\lambda_1 = \frac{q}{h} - \sqrt{h} < \frac{q}{h} + \sqrt{h} = \lambda_2$ when $h > 0$

SW is strictly hyperbolic.